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# IMPROVEMENT OF MYCIELSKI'S INEQUALITY FOR NON-NATURAL DISJOINT COVERING SYSTEMS OF Z

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If $n_i$ is a modulus of a DCS $S$, then by the Mycielski's inequality $\mathrm{card}(S) \geq 1 + \mathcal{F}(n_i)$ (see eq. (1.2) for $\mathcal{F}$) and the equality can be reached. This inequality is strengthened to $\mathrm{card}(S) \geq 6 + \mathcal{F}(n_i)$ for every non-natural DCS $S$ of the group $(Z, +)$ of integers. Moreover, if $p_3$ is the third smallest prime divisor of the common modulus of $S$, then $\mathrm{card}(S) \geq 1 + p_3 + \mathcal{F}(n_i)$. Further, (natural) DCS $S$ of $(Z, +)$ are characterized for which $\mathrm{card}(S) = 1 + \mathcal{F}(n_i)$.

## 1. Introduction and notation

The symbol $Z$ will denote the set of integers. The letter $D$ will denote the greatest common divisor and l.c.m. the least common multiple; $a \mid b$ will denote $a$ divides $b$. For integers $n > 0$, $a$ the symbol $a \pmod n$ will denote the congruence class $\{a + nx; x \in Z\}$. Although $0 \leq a < n$ is usual, arbitrary $a$ in the term $a \pmod n$ is allowed; for example $13 \pmod 7 = -1 \pmod 7 = 6 \pmod 7$.

The intersection of any two congruence classes $X = a \pmod m$, $Y = b \pmod n$ is either empty or a congruence class. The first case never takes place if $m$, $n$ are relatively prime. Further, if $X$ is a subset of $Y$ then the modulus $n$ of $Y$ divides the modulus $n$ of $X$.

The system

$$a_1 \pmod{n_1}, \quad a_2 \pmod{n_2}, \ldots, a_k \pmod{n_k} \tag{1.1}$$

will be called disjoint covering system (abbreviated: DCS) if every integer belongs to exactly one of the classes (1.1). More formally, a DCS is a partition of $Z$ into finitely many congruence classes; we always assume that these classes are given in (1.1) without repetition. The integers $n_1, \ldots, n_k$ will be called moduli of (1.1) and their least common multiple $m = \mathrm{l.c.m.}(n_1, \ldots, n_k)$ will be called the common modulus of (1.1). (The common modulus of a DCS need not be its modulus.) The trivial partition $\{Z\}$ of $Z$ will be also considered as a (degenerated) DCS.

For every positive integer $n$ denote by $Z_n$ or $Z(n)$ the partition of $Z$ into the congruence classes modulo $n$ (the symbol $Z(n)$ is sometimes used to avoid double indices). Particularly, $Z_2 = \{0 \pmod 2, 1 \pmod 2\}$ and $Z_1 = \{0 \pmod 1\} = \{Z\}$.

The number of elements of a set $X$ will be denoted card($X$). Hence if $S$ is the DCS (1.1), then $\mathrm{card}(S) = k$.

For every positive integer $n$ we denote

$$\mathscr{F}(n) = \sum_{i=1}^{r} a_i(p_i - 1), \tag{1.2}$$

if $n = \prod_{i=1}^{r} p_i^{a_i}$ is the standard form of $n$. Hence, for example, $\mathscr{F}(n) < n$ and $\mathscr{F}(m \cdot n) = \mathscr{F}(m) + \mathscr{F}(n)$ for all positive integers $m, n$. It was conjectured in [3] and proved in [6] that

$$\mathrm{card}(S) \geqslant 1 + \mathscr{F}(n_i) \tag{1.3}$$

for every DCS $S$ and any of its modulus $n_i$. It was conjectured in [7] and proved in [1] that for every DCS $S$ with the common modulus $n$

$$\mathrm{card}(S) \geqslant 1 + \mathscr{F}(n). \tag{1.4}$$

As an immediate consequence we have that equality in (1.3) is possible only if $n_i = n$; Theorem 4.1 will give a more detailed information.

A DCS $S$ will be called natural (abbreviation NDCS) if there is a finite sequence $S_0, S_1, S_2, \ldots, S_r$ of DCS such that $S_0, = Z_1$, $S_r = S$ and for every $i = 0, 1, \ldots, r-1$ there is a prime $p_i$ and a congruence class $a_i \pmod{n_i} \in S_i$ such that

$$S_{i+1} = (S_i - \{a_i \pmod{n_i}\}) \cup \{a_i + n_i x \pmod{p_i n_i}: 0 \leqslant x < p_i\}. \tag{1.5}$$

(Informally: $S_{i+1}$ arises from $S_i$ by partitioning of one of its classes into $p_i$ classes of equal moduli.) The DCS which are not natural will be called non-natural. The notion of NDCS was introduced by Porubský [4]; he also found a non-natural DCS with the common modulus 30.

The main result of the present paper will be the following improvement of (1.3) for non-natural DCS.

**Theorem 1.1.** *If $S$ is a non-natural DCS of $(Z, +)$ and $n_i$ is one of its moduli, then*

$$\mathrm{card}(S) \geqslant 6 + \mathscr{F}(n_i). \tag{1.6}$$

Hence every non-natural DCS has at least five elements more than (1.3) states. The constant 6 is the best possible, because for the DCS $S$ consisting of

$$\begin{gathered} 2 \pmod 6,\quad 4 \pmod 6,\quad 1 \pmod{10},\quad 3 \pmod{10},\quad 7 \pmod{10}, \\ 9 \pmod{10},\quad 0 \pmod{15},\quad 5 \pmod{30},\quad 6 \pmod{30},\quad 12 \pmod{30}, \\ 18 \pmod{30},\quad 24 \pmod{30},\quad 25 \pmod{30}, \end{gathered}$$

we have $\mathrm{card}(S) = 13 = 6 + 7 = 6 + \mathscr{F}(30)$.

The second section of the present paper contains some notions and results concerning so called irreducible DCS. The notion of so called Mycielski's abundance of a DCS is also introduced. It is useful in the study of improvements of (1.3). The most substantial part of the proof of Theorem 1.1 will be its proof for the case when $S$ is irreducible. This is done, in a weaker form (with 5 instead of 6), in the third section. In the fourth section the proof of Theorem 1.1 is completed. Finally, a strengthening of Theorem 1.1 is formulated which uses not only a modulus of $S$ but also its common modulus.

## 2. Irreducible DCS and splitting

We begin with some definitions and results from [2].

**Definition 2.1.** (a) Let $S_2$, $S_3$ be DCS, let $b \pmod d \in S_2$ and let $S_1$ be the DCS (1.1). We shall say that $S_3$ arises by the $b$-splitting of $S_2$ by $S_1$, and write

$$S_3 = \mathrm{Split}(S_2, b, S_1)$$
$$\text{if}\quad S_3 = (S_2 - \{b \pmod d\}) \cup \{b + a_i d \pmod{n_i d}\colon i \in \{1, \dots, k\}\}.$$

(b) For arbitrary DCS $T_1$, $T_2$, $T_3$ we shall write $\mathrm{Split}(T_1, a_1, T_2, a_2, T_3)$ instead of $\mathrm{Split}(\mathrm{Split}(T_1, a_1, T_2), a_2, T_3)$ and analogously for a greater number of splittings. Further, we define $\mathrm{Split}(S) = S$ for every DCS $S$.

The last part of this definition will be necessary e.g. in Theorem 2.4.

**Examples 2.2.** $\mathrm{Split}(Z_2, 1, Z_3)$ consists of

$$0 \pmod 2,\quad 1 \pmod 6,\quad 3 \pmod 6,\quad 5 \pmod 6,$$

and $\mathrm{Split}(Z_2, 1, Z_3, 1, Z_2)$ consists of

$$0 \pmod 2,\quad 1 \pmod{12},\quad 7 \pmod{12},\quad 3 \pmod 6,\quad 5 \pmod 6.$$

**Definition 2.3.** A DCS (1.1) will be called reducible if there is $X \subseteq \{1, \dots, k\}$, $1 < \mathrm{card}(X) < k$ such that $\bigcup\{a_i \pmod{n_i}; i \in X\}$ is a congruence class. A DCS (1.1) will be called irreducible disjoint covering system (abbreviated: IDCS) if $k > 1$ and the DCS (1.1) is not reducible.

For example, $Z_4$ is reducible because $0 \pmod 4 \cup 2 \pmod 4 = 0 \pmod 2$. The partition $Z_1$ is neither IDCS nor reducible DCS, analogously as the integer 1 is neither prime nor composite.

By induction with respect to $\mathrm{card}(S)$ we can prove:

**Theorem 2.4.** *For every DCS $S$ there are IDCS $S_1, \dots, S_r$ and integers $b_1, \dots, b_r$ such that*

$$S = \mathrm{Split}(Z_1, b_1, S_1, \dots, b_r, S_r). \tag{2.4}$$

To obtain more comprehensive notation we can extend Definition 2.1 as follows.

**Definition 2.5.** If $S_1$, $S_2$ are DCS and $X = \{b_1, \ldots, b_k\}$ is a finite set of integers such that $b_i$, $b_j$ belong to different elements of $S_1$ whenever $i \neq j$ then we shall also write $\mathrm{Split}(S_1, X, S_2)$ or $\mathrm{Split}(S_1, \{b_1, \ldots, b_k\}, S_2)$ instead of

$$\mathrm{Split}(S_1, b_1, S_2, b_2, S_2, \ldots, b_k, S_2). \tag{2.5}$$

We shall also use $\mathrm{Split}(S_1, X_1, \ldots, S_k, X_k, S_{k+1})$ analogously to 2.1.b.

A DCS $S$ is natural if and only if there are integers $a_1, \ldots, a_k$ and primes $n_1, \ldots, n_k$ such that

$$S = \mathrm{Split}(Z(1), a_1, Z(n_1), a_2, Z(n_2), \ldots, a_k, Z(n_k)).$$

The representation of a DCS $S$ in the form (2.4) is not unique. Moreover, there are NDCS which have a representation (2.4) where some of $S_j$ are non-natural IDCS.

**Theorem 2.6** (a) *If $S$ is a non-natural DCS, then its common modulus is divisible by at least three different primes, and every modulus of $S$ is divisible by at least two different primes.*

(b) *The greatest common divisor of all moduli of a non-natural IDCS is* 1.

We shall also need the following theorem from [2] it is contained also in [8] in essence.

**Theorem 2.7** *A NDCS $X$ is irreducible if and only if $X = Z(p)$ for a prime $p$.*

Now we shall give some tools for transferring of some results from the irreducible DCS to general ones.

**Theorem 2.8** *Let for DCS $S, S_1, \ldots, S_r$ (not necessarily irreducible) and integers $b_1, \ldots, b_r$ (2.4) hold. Then*

$$\mathrm{card}(S) = 1 + \sum_{j=1}^{r} (\mathrm{card}(S_j) - 1). \tag{2.8.1}$$

*Further, for every modulus $m$ of $S$ there are positive integers $m_1, \ldots, m_r$ such that*

$$m = m_1 \cdots m_r \tag{2.8.2}$$

*and for every $j = 1, \ldots, r$, the integer $m_j$ is either* 1 *or a modulus of $S_j$.*

**Proof.** We shall use induction with respect to $n$. For $n = 0$ (when $S = Z_1$) the theorem obviously holds. For the induction step in (2.8.2) it suffices to realize that every modulus of $S' = \text{split}(S, b_{n+1}, S_{n+1})$ is either a modulus of $S$ or the product of a modulus of $S$ and a modulus of $S_{n+1}$. For (2.8.1) it suffices to realize that $S'$ is obtained from $S$ by deleting of one congruence class, and adding $\text{card}(S_{n+1})$ new ones. □

**Definition 2.9** For arbitrary DCS $S$ consisting of the congruence classes (1.1) denote

$$Ab_M(S) = \min\{\text{card}(S) - 1 - \mathscr{F}(n_i); i = 1, \ldots, k\}. \tag{2.9}$$

$Ab_M(S)$ will be called the Mycielski's abundance of $S$.

Using this notion, (1.6) can be expressed in the form $Ab_M(S) \geqslant 5$.

**Theorem 2.10** *For every DCS $S$ the Mycielski's abundance $Ab_M(S)$ is nonnegative. Further, if* (2.4) *holds, then*

$$Ab_M(S) \geqslant \sum_{j=1}^{r} Ab_M(S_j). \tag{2.10}$$

**Proof.** The inequality $Ab_M(S) \geqslant 0$ is only a reformulation of (1.3). To prove (2.10) consider a modulus $m = n_i$ of $S$, and introduce the integers $m_1, \ldots, m_r$ from Theorem (2.4). If $m_j$ is a modulus of $S_j$ then we have

$$\text{card}(S_j) - 1 - \mathscr{F}(m_j) \geqslant Ab_m(S_j)$$

however, this inequality obviously holds also if $m_j = 1$, hence it holds for all $j = 1, \ldots, r$. Therefore

$$\sum_{j=1}^{r} \text{card}(S_j) - 1 - \sum_{j=1}^{r} \mathscr{F}(m_j) \geqslant \sum_{j=1}^{r} Ab_M(S_j).$$

Now by (2.8.1) and (2.8.2) we have

$$\text{card}(S) - 1 - \mathscr{F}(m) \geqslant \sum_{j=1}^{r} Ab_M(S_j).$$

The latest inequality holds for every modulus $m$ of $S$, and hence (2.10) holds. □

**Remark 2.11.** Analogously as $Ab_M$ we can define the Mycielski–Znám abundance

$$Ab_{MZ}(S) = \text{card}(S) - 1 - \mathscr{F}(m),$$

where $m$ is the common modulus of the DCS $S$. The analogon of Theorem 2.10 can be proved. Further, obviously $Ab_{MZ}(S) \leqslant Ab_M(S)$ for every DCS $S$. Examples with $<$ can be easily found. Nevertheless, we state the hypothesis $Ab_{MZ}(S) \geqslant 5$ for every non-natural DCS $S$ of the group $(Z, +)$. (I.e., (1.6) holds also if $n_i$ is replaced by the common modulus of $S$.)

## 3. The first improvement

**Lemma 3.1.** *Let $p$ be a prime, $n_1, \ldots, n_s$ be positive integers not divisible by $p$, $m$ be a positive integer, $a$, $b$, $a_1, \ldots, a_s$ be integers and let*

$$b \ (\mathrm{mod}\, p) - \bigcup_{i=1}^{s} a_i \ (\mathrm{mod}\, n_i) = a \ (\mathrm{mod}\, pm). \tag{3.1.1}$$

*Then $p \nmid m$ and*

$$b (\mathrm{mod}\, 1) - \bigcup_{i=1}^{s} a_i \ (\mathrm{mod}\, n_i) = a \ (\mathrm{mod}\, m). \tag{3.1.2}$$

**Proof.** Notice that $b\ (\mathrm{mod}\, 1) = 0\ (\mathrm{mod}\, 1) = Z$. Denote $c = a + pn_1 \cdots n_s$. Since $a \notin a_i\ (\mathrm{mod}\, n_i)$ and $c \equiv a\ (\mathrm{mod}\, n_i)$ we have $c \notin a_i\ (\mathrm{mod}\, n_i)$ for $i = 1, \ldots, s$. Since $a \in b\ (\mathrm{mod}\, p)$ and $c \equiv a\ (\mathrm{mod}\, p)$ we have $c \in b\ (\mathrm{mod}\, p)$. Therefore $c \in a\ (\mathrm{mod}\, pm)$, and hence $pm \mid c - a = pn_1 \cdots n_s$. The last formula implies $m \mid n_1 \cdots n_s$, and hence $p \nmid m$.

Now we shall show that for every $i$ the congruence classes $a_i\ (\mathrm{mod}\, n_i)$ and $a\ (\mathrm{mod}\, m)$ are disjoint. If they are not, then $p \nmid n_i$, $p \nmid m$ imply

$$b\ (\mathrm{mod}\, p) \cap a_i\ (\mathrm{mod}\, n_i) \cap a\ (\mathrm{mod}\, m) \neq \emptyset$$

which contradicts (3.1.1).

It remains to show that the set

$$Z - (a\ (\mathrm{mod}\, m) \cup \bigcup_{i=1}^{s} a_i\ (\mathrm{mod}\, n_i))$$

is empty. If not, it contains a congruence class $d\ (\mathrm{mod}\, n_1 \cdots n_s)$ (remember $m \mid n_1 \cdots n_s$). Since $p \nmid n_1 \cdots n_s$ we can arrange $d \in b\ (\mathrm{mod}\, p)$, which contradicts (3.1.1). □

Notice that $a_i\ (\mathrm{mod}\, n_i)$, $i = 1, \ldots, s$ need not be disjoint in Lemma 3.1. Hence this lemma can be useful also for covering systems of $Z$. We shall reformulate this lemma also for cyclic groups.

**Lemma 3.2.** *Let $G$ be a cyclic group, $H$ its maximal subgroup, $F$ be a subgroup of $G$, let $G_1, \ldots, G_s$ be subgroups of $G$ which are not contained in $H$, let $a$, $a_1, \ldots, a_n$, $b$ be elements of $G$ and*

$$bH - \bigcup_{i=1}^{s} a_i G_i = aF. \tag{3.2.1}$$

*Then there is a subgroup $F_1$ of $G$ which is not contained in $F$ such that $F = F_1 \cap H$*

and

$$G - \bigcup_{i=1}^{s} a_i G_i = aF_1. \tag{3.2.2}$$

Since the transformation of 3.1 to 3.2 is straightforward (for example, $p$, $n_i$ correspond to the indices $[G:H]$, $[G:G_i]$), the proof will be omitted. The following example shows that the word "cyclic" can be neither omitted nor replaced by "finite abelian".

**Example 3.3.** Let $G = Z_2^3$, $H = Z_2^2 \times \{0\}$, $G_1 = \{(0,0,0), (0,0,1)\}$, $G_2 = \{(0,0,0), (0,1,1)\}$ and $F = \{(0,0,0), (1,1,0)\}$. Then

$$H - ((0,1,1)G_1 \cup (1,1,1)G_2) = \{(0,0,0), (1,1,0)\}$$

is a subgroup of $H$. However

$$G - ((0,1,1)G_1 \cup (1,1,1)G_2) = \{(0,0,0), (1,1,0), (0,0,1), (1,0,1)\}$$

is not a coset of $G$ by any its subgroup. (Since it contains $(0,0,0)$ it suffices to realize that it is not a subgroup of $G$.)

**Lemma 3.4.** *If $S$ is a non-natural IDCS, then $Ab_M(S) \geq 4$.*

**Proof.** Let $n_i$ be a modulus of $S$. We have to prove

$$\mathrm{card}(S) \geq 5 + \mathcal{F}(n_i). \tag{3.4}$$

Let $n$ be the common modulus of $S$ and $p$ be its greatest prime divisor. Since $n$ is divisible by at least three primes we have $p \geq 5$. Assume that $S$ is (1.1), and that $p \nmid n_i$ for $i = 1, \ldots, s$ and $p \mid n_i$ for $i = s+1, \ldots, k$. Since $S$ is irreducible we have $s > 1$ and since $p \mid n$ we have $s < k$. We may also assume $a_i = 0$ without loss of generality.

Consider the partition of $Z$ into $p$ congruence classes $b \pmod p$, $b = 0, 1, \ldots, p-1$. Since $p \mid n$ at least one of these classes contains a member of $S$. Then each of them must contain a member of $S$ (see e.g. [2]). However, if a class $b \pmod p$ contains exactly one member $a_j \pmod{n_j}$ of $S$, then by Lemma 3.1 the congruence classes

$$a_1 \pmod{n_1}, \ldots, a_s \pmod{n_s}, a_j \left(\mathrm{mod}\ \frac{n_j}{p}\right)$$

form a DCS,

$$a_j \left(\mathrm{mod}\ \frac{n_j}{p}\right) = \bigcup_{i=s+1}^{k} a_i \pmod{n_i}$$

and hence $S$ would not be irreducible. Therefore every class $b \pmod p$ contains at least two members of $S$. Therefore there are at least $2 \cdot (p-1)$ members of $S$ which are disjoint with $0 \pmod p$.

Now consider $0 \pmod p$ as the additive group. The nonempty intersections of the members of $S$ with $0 \pmod p$ form a DCS $S'$ of $0 \pmod p$, and by the above considerations

$$\operatorname{card}(S) \geqslant \operatorname{card}(S') + 2 \cdot (p-1).$$

Since $A = a_i \pmod{n_i} \cap 0 \pmod p \neq \emptyset$ we have $A \in S'$, and the index $n_i'$ of $A$ in $0$ $(\bmod p)$ is $n_i$ if $p \nmid n_i$ and $n_i/p$ if $p \mid n_i$. In both cases we have $\operatorname{card}(S') \geqslant 1 + \mathcal{F}(n_i')$, $\mathcal{F}(n_i') \geqslant \mathcal{F}(n_i) - \mathcal{F}(p) = \mathcal{F}(n_i) - (p-1)$, and hence

$$\operatorname{card}(S) \geqslant \mathcal{F}(n_i) + p$$

However, $p \geqslant 5$, and hence (3.4) holds. □

## 4. Proof of the main theorem

The first theorem of this part is necessary for the proof of Theorem 1.1 but it is also self-interested. It characterizes the DCS with the property $Ab_M(S) = 0$.

**Theorem 4.1.** *Let $S$ be a DCS and let $a_i \pmod{n_i}$ be one of its congruence classes. Then the following conditions are equivalent*:

(i) $\operatorname{card}(S) = 1 + \mathcal{F}(n_i)$;

(ii) *There is a finite sequence of primes $p_1, \ldots, p_r$ such that $n_i = p_1 \cdots p_r$ and* $S = \operatorname{Split}(Z_1, a_i, Z(p_1), a_i, Z(p_2), \ldots, a_i, Z(p_r))$.

*Further, if* (i), (ii) *hold, then $n_i$ is the common modulus of $S$.*

**Proof.** Let $\operatorname{card}(S) = 1 + \mathcal{F}(n_i)$, and let $m$ be the common modulus of $S$. Obviously $n_i \mid m$. If $n_i \neq m$, then $\mathcal{F}(n_i) < \mathcal{F}(m)$ and $\operatorname{card}(S) < 1 + \mathcal{F}(m)$, which contradicts (1.4). Therefore $m = n_i$. Now let $S$ be represented in the form (2.4) with irreducible $S_1, \ldots, S_r$, and let $m_1, \ldots, m_r$ be the integers from Theorem 2.8. Since $Ab_M(S) = 0$, Theorem 2.10 implies $Ab_M(S_j) = 0$ for all $j = 1, \ldots, r$. Therefore every $S_j$ is natural (by Lemma 3.4), and hence $S_j = Z(p_j)$ for a prime $p_j$ (by Lemma 2.7). So we obtain

$$S = \operatorname{Split}(Z_1, b_1, Z(p_1), \ldots, b_r, Z(p_r)).$$

Now denote by $q_j$ the modulus of this element of $S_j' = \operatorname{Split}(Z_1, b_1, Z(p_1), \ldots, b_j, Z(p_j))$ which contains $a_i$. For every $j = 1, \ldots, r$ we have either $q_j = q_{j-1} p_j$ or $q_j = q_{j-1}$. However, $q_0 = 1$ and $q_r = p_1 \cdots p_r$, hence $q_j = q_{j-1}$ for no $j = 1, \ldots, r$. Hence by each step from $S_{j-1}'$ to $S_j'$ the congruence class containing $a_i$ is splitted. So every $b_j$ can be replaced by $a_i$, and (4.1) is obtained. Conversely, let (4.1) hold. Denote

$$T_j = \operatorname{Split}(Z_1, a_i, Z(p_1), \ldots, a_i, Z(p_j))$$

for every $j = 0, \ldots, r$; hence $T_0 = Z_1$, and $T_r = S$. Then $\operatorname{card}(T_j) = 1 +$

$\mathcal{F}(p_1 \cdots p_j)$ can be easily proved by induction with respect to $j$. For $j = r$ we obtain (i). □

**Lemma 4.2.** *Let $S$ be a DCS such that $Ab_M(S) = 0$. Let $A$, $B$ be two disjoint congruence classes such that $A \cup B$ is not a congruence class, and $A \cup B$ is the union of a subset of $S$. Then both $A$ and $B$ are the unions of subsets of $S$.*

**Proof.** If $Ab_M(S) = 0$, then $\mathrm{card}(S) = 1 + \mathcal{F}(n_i)$ for a modulus $n_i$ of $S$, and $S$ can be represented in the form (4.1). Consider the set of all congruence classes of all DCS $T_j$, $j = 0, \ldots, r$ from the proof of Theorem 4.1, and its partial order by the set-theoretical inclusion $\subseteq$. The Haase diagram of this partial ordered set is a rooted tree. The leaves of this tree are exactly members of $S = T_r$ (of course, they may belong also to some $T_j$, $j < r$). The other vertices form a finite chain. Let $C$ be the smallest element of this chain which contains $A \cup B$. Since $A \cup B$ is not a congruence class, $A \cup B$ has nonempty intersection with at least two sons of $C$. At least one of these sons is a leaf of the considered tree; denote it by $D$. Obviously $D \subseteq A \cup B$. Further, let $M$ be the subset of $S$ such that $A \cup B = \bigcup M$.

Let $m$ be the modulus of $C$ and $pm$ (where $p$ is a suitable prime) be the moduli of its sons. Since $A \subset C$, $B \subset C$ the moduli of $A$, $B$ are $km$, $lm$ for suitable integers $k$, $l$.

If $A \not\subseteq D$, $B \not\subseteq D$, then $p \nmid k$, $p \nmid l$. However, in this case the moduli of $A \cap D$, $B \cap D$ are $pkm$, $plm$, respectively, and since $A \cup B \subset C$ and $(A \cap D) \cup (B \cap D) = D$ we have

$$\frac{1}{km} + \frac{1}{lm} < \frac{1}{m}, \qquad \frac{1}{pkm} + \frac{1}{plm} = \frac{1}{pm},$$

which is a contradiction. Therefore $A \subseteq D$ or $B \subseteq D$. Further assume $A \subseteq D$, $B \not\subseteq D$ (both inclusions cannot hold).

If $B \cap D \neq \emptyset$ then since $D = A \cup (B \cap D)$ the moduli of both $A$, $B \cap D$ are equal $2pm$. Consequently, $B$ has the modulus $2m$. Hence $p \neq 2$ (otherwise $B = D$). But then we have at least one more leaf $D_1$ with the modulus $pm$. We have $\emptyset \neq B \cap D_1 \subset D_1$, $A \cap D_1 = \emptyset$, hence $\emptyset \neq (A \cup B) \cap D_1 \subset D_1$ which is a contradiction.

Finally, if $A \subseteq D$ and $B \cap D = \emptyset$, then $A = D$ and hence $A = \bigcup\{D\}$, $B = \bigcup(M - \{D\})$. □

**Remark 4.3.** The condition $Ab_M(S) = 0$ cannot be replaced by "$S$ is natural" in Lemma 4.2. To show that it suffices to consider $S = \{0 \pmod{15}\} \cup \{i \pmod{30};\ 1 \leqslant |i| \leqslant 14\}$ and $A = 0 \pmod 6$, $B = 5 \pmod{10}$.

**Lemma 4.4.** *If $A$, $B$, $C$, $D$ are congruence classes, $A \cap B = \emptyset$, $C \cap D = \emptyset$ and $A \cup B = C \cup D$, then $\{A, B\} = \{C, D\}$.*

**Proof.** Denote $X=A\cap C$, $Y=A\cap D$, $Z=B\cap C$, $T=B\cap D$. Each of the sets $X, Y, Z, T$ is either empty or a congruence class. Further, $A=X\cup Y$, $B=Z\cup T$, $C=X\cup Z$, $D=Y\cup T$. Since $A\neq\emptyset$ we have $X\neq\emptyset$ or $Y\neq\emptyset$, and analogously for $B, C, D$. If $X\neq\emptyset$, $Y\neq\emptyset$, then the moduli $x, y$ of $X, Y$ are multiples of the modulus $n$ of $A$ and $1/x+1/y=1/n$, which easily implies $x=y=2n$. Analogously, if $Y, Z$ are nonempty then their moduli are equal, etc.

Let at least three of the sets $X, Y, Z, T$ be non-empty; for example let $X\neq\emptyset$, $Y\neq\emptyset$, $Z\neq\emptyset$. If $m, n$ are moduli of $A, C$, then the modulus of $X$ must be simultaneously $2m$ and $2n$, hence $m=n$. Therefore, $A, C\in Z(m)$, and since $A\cap C=X\neq\emptyset$ we have $A=C=X$, which is a contradiction. Hence at most two of $X, Y, Z, T$ are non-empty.

We can easily see that either $X\neq\emptyset$, $T\neq\emptyset$, or $Y\neq\emptyset$, $Z\neq\emptyset$. In the first case we have $A=C$, $B=D$ and in the other one $A=D$, $B=C$. Hence in both cases $\{A, B\}=\{C, D\}$. □

**Lemma 4.5.** *Let $p>2$ be a prime, $n_1, \ldots, n_s$ be positive integers not divisible by $p$, $m_1, m_2$ be positive integers, $a_1, \ldots, a_s b, c_1, c_2$ be integers, $c_1 \,(\mathrm{mod}\, pm_1)\cap c_2\,(\mathrm{mod}\, pm_2)=\emptyset$ and*

$$b\,(\mathrm{mod}\, p)-\bigcup_{i=1}^{s} a_i\,(\mathrm{mod}\, n_i)=c_1\,(\mathrm{mod}\, pm_1)\cup c_2\,(\mathrm{mod}\, pm_2). \tag{4.5.1}$$

*Then $p\nmid m_1$, $p\nmid m_2$, $c_1\,(\mathrm{mod}\, m_1)\cap c_2\,(\mathrm{mod}\, m_2)=\emptyset$ and*

$$b\,(\mathrm{mod}\, 1)-\bigcup_{i=1}^{s} a_i\,(\mathrm{mod}\, n_i)=c_1\,(\mathrm{mod}\, m_1)\cup c_2\,(\mathrm{mod}\, m_2). \tag{4.5.2}$$

**Proof.** Denote the least common multiple of $n_1, \ldots, n_s$ by $n$. The left side of (4.5.1) is a union of several congruence classes modulo $pn$. Now distinguish two cases. If one of these classes has nonempty intersections with both $c_1\,(\mathrm{mod}\, pm_1)$, $c_2\,(\mathrm{mod}\, pm_2)$, then these intersections are congruence classes of the form

$$d_1\,(\mathrm{mod}\, pm_1e_1), \qquad d_2\,(\mathrm{mod}\, pm_2e_2).$$

Since their union is a congruence class modulo $pn$ we have $pm_1e_1=pm_2e_2=2pn$, and then, obviously, $p\nmid m_1$, $p\nmid m_2$. In the opposite case each of the congruence classes $c_i\,(\mathrm{mod}\, pm_i)$, $i=1, 2$, contains a congruence class modulo $pn$. Hence $pm_i \mid pn$, and we also have $p\nmid m_i$ for $i=1, 2$.

If $c_1\,(\mathrm{mod}\, m_1)\cap c_2\,(\mathrm{mod}\, m_2)\neq\emptyset$, then there is an integer $d$ such that for all integers $e$

$$d+em_1m_2\in c_1\,(\mathrm{mod}\, m_1)\cap c_1\,(\mathrm{mod}\, m_2).$$

Since $p\nmid m_1m_2$ the integer $e$ can be chosen so that $p\mid d+em_1m_2$, and since

$p \mid c_1$, $p \mid c_2$ we have

$$d + em_1m_2 \in c_1 \,(\mathrm{mod}\, pm_1) \cap c_2 \,(\mathrm{mod}\, pm_2)$$

which is a contradiction.

The proof of (4.5.2) is now very similar to the proof of (3.1.2) and therefore it is omitted. □

**Lemma 4.6.** *If S is a non-natural IDCS, then* $Ab_M(S) \geqslant 5$.

**Proof.** Compare Lemma 4.6 with Lemma 3.4. The only difference is that we have 5 instead of 4, now. Hence we must only exclude the equality in (3.4). Introduce all notations from the proof of Lemma 3.4. If $\mathrm{card}(S) = 5 + \mathcal{F}(n_i)$, then

(i) $\mathrm{card}(S') = 1 + \mathcal{F}(n_i')$;
(ii) Every class $b \,(\mathrm{mod}\, p)$, $1 \leqslant b \leqslant p-1$ contains exactly two members of $S$;
(iii) $p \nmid n/n_i$.

Lemma 4.5 implies that for every $b = 1, \ldots, p-1$ there are two disjoint congruence classes $X_b$, $Y_b$ with the moduli not divisible by $p$ such that

$$X_b \cap b \,(\mathrm{mod}\, p), \qquad Y_b \cap b \,(\mathrm{mod}\, p)$$

are the members of $S$ contained in $b \,(\mathrm{mod}\, p)$. Further, by (4.5.2) we have

$$X_b \cup Y_b = Z - \bigcup_{i=1}^{s} a_i \,(\mathrm{mod}\, n_i).$$

Therefore $X_b \cup Y_b$ does not depend on $b$, hence $X_b \cup Y_b = X_1 \cup Y_1$. Then Lemma 4.4 implies $\{X_b, Y_b\} = \{X_1, Y_1\}$, and we may assume $X_b = Y_1$, $Y_b = Y_1$. Now consider the congruence classes

$$A = X_1 \cap 0 \,(\mathrm{mod}\, p), \qquad B = Y_1 \cup 0 \,(\mathrm{mod}\, p).$$

The set $A \cup B$ is the union of a proper subset of $S$. Since $S$ is irreducible, $A \cup B$ is not a congruence class. We shall apply Lemma 4.2 and Theorem 4.1 to the DCS $S'$ of $0 \,(\mathrm{mod}\, p)$. (To be quite precise, we have to apply them to the DCS $S''$ of $Z$ which is induced from $S'$ by the isomorphism $x \mapsto x/p$ of the additive group $0 \,(\mathrm{mod}\, p)$ onto $Z$.) Then we obtain that $A$, $B$ are unions of some proper subsets of $S'$. Hence $X$, $Y$ are unions of some proper subsets of $S$. However, $S$ is irreducible, which is a contradiction. □

**Proof of Theorem 1.1.** Represent the DCS $S$ in the form (2.4) with irreducible $S_1, \ldots, S_r$. At least one of $S_j$ is non-natural, and hence $Ab_M(S_j) \geqslant 5$ for this $S_j$. Now Theorem 2.10 implies $Ab_M(S) \geqslant 5$, which immediately gives (1.6). □

**Remark 4.7.** Theorem 2.10 and Lemma 4.6 also make possible to estimate the number of non-natural DCS $S_j$ in any representation (2.4) of a DCS $S$. This number is at most $\lfloor\frac{1}{5}Ab_M(S)\rfloor$.

The right-hand side of (1.6) in Theorem 1.1 does not depend on the common modulus of $S$. If we allow such a dependence we can formulate a stronger result:

**Theorem 4.8.** *Let $S$ be a non-natural DCS, let $p_1 < p_2 < \cdots < p_k$ be the list of all prime divisors of the common modulus of $S$ and let $n_i$ be a modulus of $S$. Then*

$$\mathrm{card}(S) \geqslant 1 + p_3 + \mathcal{F}(n_i).$$

The inequality $k \geqslant 3$ (and hence, the existence of $p_3$) follows from Theorem 2.6(a). In Lemma 3.4 we may replace 5 by $p_3$ because $p \geqslant p_3$. The proof need not be changed in essence.

## References

[1] I. Korec, On a generalization of Mycielski's and Znám's conjectures about coset decomposition of Abelian groups, Fund. Math. 85 (1974) 41–48.

[2] I. Korec, Irreducible disjoint covering systems, Acta Arithmetics 44 (1984) 389–395.

[3] J. Mycielski and W. Sierpinski, Sur une propriété des ensembles linéaires, Fund. Math. 58 (1966) 143–147.

[4] Š. Porubský, Natural exactly covering systems of congruences, Czech. Math. J. 24(1974) 598–606.

[5] Š. Porubský, Results and problems on covering systems of residue classes, Mitt. Math. Semin. Giessen 150 (1981) 1–85.

[6] Š. Znám, A remark to a problem of J. Mycielski on arithmetic sequences, Coll. Math. 20 (1969) 67–70.

[7] Š. Znám, On exactly covering systems of arithmetic sequences, Colloquia Mathematica Soc. J. Bolyai 2, Number Theory (1968) 221–224.

[8] Š. Znám, A survey of covering systems of congruences, Acta Math. Univ. Comen. 40–41 (1982) 59–79.